Let $\{X_i\}$ iid $L^2(\mathbb{P})$. $s_n^2 = \sum_{i=1}^{n} \sigma_i^2$ $\mu_n = E[X_n]$ $\sigma_n^2 = Var(X_n)$

If $s_n^2 \to \infty$ and

$$\lim_{n\to\infty} \frac{1}{s_n^2} \mathbb{E}\left[(X_j - \mu_j)^2 \mathbb{1}_{\{|X_j - \mu_j| > \epsilon s_n\}} \right] = 0 \quad \forall \epsilon > 0.$$

Then $\dfrac{T_n - \sum_{i=1}^{n} \mu_i}{s_n} \xrightarrow{d} N(0,1)$ where $T_n = \sum_{i=1}^{n} X_i$

This is the Lindeberg-Feller theorem.

Replace X_i by $X_i - \mu_i$ ie assume they are mean 0.

Truncate $X_i' = X_i \mathbb{1}_{\{|X_i| < \epsilon s_n\}}$ $T_n' = \sum_{i=1}^{n} X_i'$ $\mu_n' = ET_n'$

$s_n'^2 = Var\, T_n'$

Recall $\left| Ef\left(\frac{T_n' - \mu_n'}{s_n'^2}\right) - Ef(N(0,1)) \right| \leq \frac{2C}{s_n'^2} \sum_{i=1}^{n} E|X_i'|^3$ (7.71 in Khoshnevisan)

(Lindeberg replacement bound)

$E|X_i'|^3 \leq \epsilon s_n E|X_i'|^2 \leq \epsilon s_n E|X_i^2| = \epsilon s_n \sigma_i^2$

$\Rightarrow \left| Ef\left(\frac{T_n' - \mu_n'}{s_n'^2}\right) - Ef(N(0,1)) \right| \leq \frac{2C\epsilon s_n^2}{s_n'^2}$

$$E\left[\left|g\left(\frac{T_n'-\mu_n'}{s_n}\right)-g\left(\frac{T_n}{s_n}\right)\right|\right] \le \|g'\|_\infty\, E\left|\frac{T_n'-\mu_n'-T_n}{s_n}\right|^{2}$$

$$\le \|g'\|_\infty \frac{\sqrt{Var(T_n'-T_n)}}{s_n}$$

Jensen

$$\frac{Var(T_n'-T_n)}{s_n^2} = \frac{1}{s_n^2}\sum_{i=1}^{n} Var\left(X_i 1_{\{|X_i|>\epsilon s_n\}}\right)$$

$$\le \frac{1}{s_n^2}\sum_{i=1}^{n} E\left[X_i^2 1_{\{|X_i|>\epsilon s_n\}}\right] \longrightarrow 0$$

$$Var(X_i') = Var\left(X_i 1_{\{X_i\le\epsilon s_n\}}\right) = E\left[X_i^2 1_{\{X_i\le\epsilon s_n\}}\right] - E\left[X_i 1_{\{X_i\le\epsilon s_n\}}\right]^2$$

$$= Var(X_i) - E\left[X_i^2 1_{\{X_i>\epsilon s_n\}}\right] - \left(E\left[X_i 1_{\{X_i>\epsilon s_n\}}\right]\right)^2 - 2\cancel{E[X_i]}^{\,0}\, E\left[X_i 1_{\{X_i>\epsilon s_n\}}\right]$$

by mean 0 assumption.

$$\left(E\left[X_i 1_{\{X_i>\epsilon s_n\}}\right]\right)^2 \le E\left[X_i^2 1_{\{X_i>\epsilon s_n\}}\right].$$

$$\Rightarrow\quad s_n'^2 = s_n^2 + O\left(\sum_{i=1}^{n} E\left[X_i^2 1_{\{X_i>\epsilon s_n\}}\right]\right)$$

$$\Rightarrow\quad \frac{s_n'^2}{s_n^2} \longrightarrow 1 \quad \text{(by Lindeberg condition)}$$

Putting these two together completes the proof.